Notes for STAT 533

Sep 9 2021
Continue from Poisson Processes
We know that $((S_1, \ldots, S_n)|N(t) = n) \sim (U_{(1)}, \ldots U_{(n)})$ where $(U_{(1)}, \ldots)$ are the order statistics of $U_t \sim U(0,t)$ for S_i as the ith occurrence time in Poisson processes.
Now classify two events:

- Type-I $P(\text{an event is Type I}) = p(s)$ where s is the arrival time of the event.
- Type-II $P(\text{an event is Type II}) = 1 - p(s)$.

$N_i(t)$ is the number of type-i events that occur by time t and $i = 1, 2$.
conclusion : $N_1(t)$ and $N_2(t)$ are independent Poisson rvs having respective means λtp and $\lambda t(1-p)$ where $p = \frac{1}{t}\int_0^t p(s)ds$
$N(t) = N_1(t) + N_2(t)$
proof we can have

$$\begin{aligned} P(N_1(t) = n, N_2(t) = m) &= \sum_{k=0}^{\infty} P(N_1(t) = n, N_2(t) = m|N(t) = k)P(N(t) = k) \\ &= P(N_1(t) = n, N_2(t) = m|N(t) = n+m)P(N(t) = n+m) \end{aligned} \tag{0.1}$$

If an event occurs in $[0, t]$, then

$$P(\text{this event is Type I}) = p = \frac{1}{t}\int_0^t p(s)ds$$

This is because $P(\text{Type I}|S = s \le t) = p(s)$ and $S|S \le t \sim U(0,t)$.
Thus the first term in the above equation just gives the Bernoulli for $n+m$ trials with n success with success rate p. Thus, we have

$$\begin{aligned} (N_1(t) = n, N_2(t) = m) &= \binom{n+m}{n} p^n (1-p)^m \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!} \\ &= e^{-\lambda tp}\frac{(\lambda tp)^n}{n!} e^{-\lambda t(1-p)}\frac{[\lambda t(1-p)]^m}{m!} \end{aligned}$$

$M/G/\infty$ **problem**
$N_1(t)$ is the number of customers that have completed their services by time t. $N_2(t)$ is the number of customers that are in service at time t. What is the joint distribution of $(N_1(t), N_2(t))$?
If a customer arrives at time s and $s \le t$, what is the probability that $P(\text{type I}|s \le t)$?

this is just $P(\text{service time} \leq t-s) = G(t-s) = p(s)$. and thus

$$P(\text{Type I}) = \frac{1}{t}\int_0^t p(s)ds = \frac{1}{t}\int_0^t G(t-s)ds = \frac{1}{t}\int_0^t G(s)ds$$

Non-homogenous Poisson Process Definition: The conting process $N(t)$ is saied to be a non-stationary or non-homogeneous Poisson process with intensity function $\lambda(t)$ $t \geq 0$ if

1. N(0) = 0
2. $\{N(t), t \geq 0\}$ has independent increment
3. $P(N(t+h) - N(t) \geq 2) = o(h)$
4. $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$

If $\lambda(t) = \lambda$, this boils down to Poisson process.
Let $m(t) = \int_0^t \lambda(s)ds$
Claim: $p(N(t+s) - N(t) = n) = \exp(-(m(t+s) - m(t)))\dfrac{(m(t+s) - m(s))^n}{n!}$ or $N(t+s) - N(t) \sim$ poisson$(m(t+s) - m(t))$
sketch of proof Fix t and define $p_n(s) = p(N(t+s) - N(t) = n)$. Then we have

$$\begin{aligned} p_0(s+h) &= P(N(t+s+h) - N(t) = 0) \\ &= P(0 \text{ events in } (t, t+s],\ 0 \text{ events in } (t+s, t+s+h]) \\ &= P(0 \text{ events in} (t, t+s])P(0 \text{ events in } (t+s, t+s+h] \\ &= P_0(s)[1 - \lambda(t+s)h + o(h)] \end{aligned}$$

and this implies that

$$\frac{P_0(s+h) - P_0(s)}{h} = \lambda(t+s)P_0(s) + \frac{o(h)}{h} \quad \Rightarrow \quad P_0'(s) = -\lambda(t+s)P_0(s)$$

As a result, $P_0(s) = e^{-[m(t+s) - m(t)]}$
Let $X(t) \sim PP(\lambda)$ and $Y(t) \sim$ NH-PP$(\lambda(t))$ and $\lambda(t) \leq \lambda$, for $t \geq 0$. **Claim: we can generate $Y(t)$ as a random sample from a homogeneous $PP(\lambda)$ process $X(t)$** as:
Event of $X(t)$ occurs at time t is counted with probability $\frac{\lambda(t)}{\lambda}$
proof: $P(y(t+h) - y(t) = 1) = P(X(t+h) - X(t) = 1) \cdot \frac{\lambda(t)}{\lambda} = \lambda(t)h + o(h)$ and this is the definition of the NH-PP$(\lambda(t))$.
Example: The output process of $M/G/\infty$ is a NH-PP with $\lambda(t) = \lambda G(t)$.
we would check the following to properties to show the claim.

1. The number of departures in $(s, s+t]$ is Poisson$(\lambda \int_s^{s+t} G(y)dy)$
2. the number of departures in disjoint time intervals are independent.

consider time interval $(s, s+t]$. We have two scenarios: a. arrive before s and departs between $(s, s+t]$; b. arrive after s and leave before $s+t$.
We call an arrival Type I if it departures in $(s, s+t]$. Then we have

$$P(\text{Type I}|S = y < s+t) = p(y) = \begin{cases} G(t+s-y) - G(s-y) & \text{if } y \le s \\ G(s+t-y) & \text{if } s < y < s+t \end{cases}$$

The number of departures will be Poisson distributed with mean $\lambda(s+t)p$ with

$$p = \frac{1}{t+s}\int_0^{s+t} p(y)dy = \lambda \int_s^{s+t} G(y)dy$$

To prove the independence for disjoint intervals we can define Type I ad departs in interval 1 and Type II departs in interval 2 and Type III otherwise.

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Renewal Process
Definition: Renewal process is a counting process with interarrival times i.i.d $F(t)$.
Let $\{X_n, n = 1, 2, \cdots\}$ are i.i.d. with common distribution F with $F(X_n = 0) < 1$ and thus $\mu = E[X_n] > 0$.
Let $S_0 = 0$, we define

$$S_n = \sum_{i=1}^{n} X_i \text{ with } n \ge 1$$
$$N(t) = \sup\{n | S_n \le t\}.$$

Some questions:
Q1: Can $N(t) = \infty$?
Ans: No, because $\frac{S_n}{n} \to \mu$ thus $S_n \to \infty$. As a result, S_n can be less than or equal to t for at most a finite number of values of n. $\Rightarrow N(t)$ must be finite.
Q2: Distribution of $N(t)$?
Ans: $\{N(t) \ge n\} \quad \equiv \quad \{S_n \le t\}$. As a result,

$$\begin{aligned} P(N(t) = n) &= P(N(t) \ge n) - P(N(t) \ge n+1 \\ &= P(S_n \le t) - P(S_{n+1} \le t) \end{aligned}$$

Thus, we need to calculate the distribution of the sum of X_i. We need to use convolution.
For $X \sim F$, $y \sim G$. We say $X + Y \sim F * G$, the convolution of F and G. defined as

$$(F * G)(a) = \int_{-\infty}^{\infty} P(X + Y \le a | Y = y) dG(y) = \int_{-\infty}^{\infty} F(a-y) dG(y)$$

Then define F_n recursively as $F_n = F * F_{n-1}$. As a result,

$$P(N(t) = n) = F_n(t) - F_{n+1}(t).$$

Let $m(t) = E[N(t)]$, the renewal function. We claim $m(t) = \sum_{n=1}^{\infty} F_n(t)$.

proof: We have $N(t) = \sum_{n=1}^{\infty} I_n$ where I_n indicates if n-th renewal occurred in $[0, t]$. Thus,

$$E[N(t)] = \sum_{n=1}^{\infty} E[I_n] = \sum_{n=1}^{\infty} P(S_n \leq t) = \sum_{n=1}^{\infty} F_n(t).$$

Claim: $m(t) < \infty$ for any finite time t. which should be obvious because $N(t) < \infty$ and $m(t) = E[N(t)]$.
Another proof: Since $p(X_n = 0) < 1$ there exists an $\alpha > 0$ wuch that $P(X_n \geq \alpha) > 0$.
Define $\bar{X}_n = \alpha I(X_n \geq \alpha) \leq X_n$. Then define $\bar{N}(t) = \sup\{n|\bar{X}_1 + \bar{X}_2 + \cdots\}$ As a result, $\bar{N}(t) \geq N(t)$. Thus, $E[\bar{N}(t)] \geq E[N(t)]$. Then we can try to prove $E[\bar{N}(t)] < \infty$.
Define $G_i :$ number of X_j needed to get the first α. Thus, G_1 is geometric with success rate $p = P\{X_1 \geq \alpha\}$. Thus $E[G_1] = 1/p$. Because the X_i are indpendent, we have

$$E[\bar{N}(t)] \leq \frac{t/\alpha + 1}{p(X_1 \geq \alpha)}$$

Define $N(\infty) = \lim_{t\to\infty} N(t)$. $\{N(\infty) < \infty\} = \{\text{one of the interarrival times } = \infty\}$
Thus, $P(N(\infty) < \infty) = P(X_n = \infty \text{ for some } n) \leq \sum_{n=1}^{\infty} P(X_n = \infty) = 0$ Thus, $N(\infty) = \infty$ with probability 1.
Proposition: $\frac{N(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$ with probability 1.
Now notice $S_{N(t)}$ is the time of the last renewal prior to or at time t. And $S_{N(t)+1}$ is the time of the first renewal after time t. Now we can prove the proposition.
proof: Since $S_{N(t)} \leq t < S_{N(t)+1}$ and thus

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

Because $N(t) \to \infty$ and thus as $t \to \infty$, $S_{N(t)}/N(t) \to \mu$. Also,

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1}\frac{N(t)+1}{N(t)} \to \frac{S_{N(t)+1}}{N(t)+1} \to \mu$$

As a result, $N(t)/t \to 1/\mu$ when $t \to \infty$.

Wald's Equation
Define **stopping time** as an integer-valued r.v. N for the sequence X_i if $\{N = n\}$ is independent of $(X_{n+1}, X_{n+2}, \dots)$
Example: X_n i.i.d. and $P(X_n = 0) = \frac{1}{2} = P(X_n = 1)$ and $N = \min\{n|X_1 + X_2 + \cdots + X_n = 10\}$ will be a stopping time.
Wald's equation: If $X_1, X_2, \dots$ are i.i.d. with finite expectations, and if N is a stopping time for X_t such that $E[N] < \infty$ then $E[\sum_{n=1}^{N} X_n] = E[N]E[X]$.
proof: Notice, since N depends on X_i, we cannot use conditional probability to get the result. Use Indicator $I_n = 1_{\{N\geq n\}}$ and thus,

$$\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i I_i \quad \Rightarrow \quad E[\sum_{i=1}^{N} X_i] = \sum_{i=1}^{\infty} E[X_i I_i]$$

Notice I_n indicates $\{N \geq n\}$ which is $\{N \leq n-1\}^c$ which means I_n is independent of X_n Thus.

$$E_{i=1}^{N} X_i = \sum_{i=1}^{\infty} E[X_i]E[I_i] = E[X] \sum_{i=1}^{\infty} P\{N \geq n\} = E[X]E[N]$$

where we assumed $E[N]$ is finite so that we can exchange sum and expectation.

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Example for Wald's equation: Let $N = \min\{n | X_1 + \cdots + X_n = 10\}$ and $P\{X_1 = 0\} = P\{X_1 = 1\} = \frac{1}{2}$. Then what is $E(N)$?

We know that $E[X_1 + \cdots + X_N] = 10$ implies that $E[X_1]E[N] = 10$ which shows $E[N] = 20$. But we can calculate the distribution of N:

$$P(N = a) = \binom{n-1}{9} \frac{1}{2}^n$$

But if $P(X_1 = -1) = 1/2$ and $P(X_1 = 1) = 1/2$ and we want to know $N = \min\{n | X_1 + \cdots + X_n = 1\}$ and we get

$$E[N]E[X_1] = 1 \quad \Rightarrow \quad E[N] \times 0 = 1$$

this cannot be right. Because $E[N] = \infty$, the Wald's equation assumption does not hold.

Now back to renewal theory.

Notice $N(t) + 1$ is a stopping time. because $N(t) + 1 = n$ is indpendent of $X_{n+1}, X_{n+2}, \cdots$.

Since $E[X_1] < \infty$, we have

$$E[X_1 + \cdots + X_{N(t)+1}] = E[X]E[N(t) + 1]$$

Thus, $E[S_{N(t)+1}] = \mu(m(t) + 1)$

Theorem The elemetary Renewal theorem:

$$\frac{m(t)}{t} \to \frac{1}{\mu} \text{ as } t \to \infty$$

proof: Because $S_{N(t)+1} > t$, take expectation, we have $\mu(m(t) + 1) > t$ thus, we have

$$\frac{m(t)}{t} + \frac{1}{t} > \frac{1}{\mu}$$

but when $t \to \infty$, we have

$$\liminf_{t \to \infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

. For a positive M, define$\bar{X}_n$ for $n = 1, \ldots,$ so that

$$\bar{X}_n = \begin{cases} X_n & \text{if } X_n \leq M \\ M & \text{if } X_n > M \end{cases}$$

Now, define $\bar{S}_n = \sum_{i=1}^{n} \bar{X}_i$ and $\bar{N}_t = \sup\{n | \bar{S}_n \leq t\}$. Notice because $\bar{X}_{\bar{N}(t)} \leq M$, and $\bar{S}_n \leq t$, we have $\bar{S}_{\bar{N}(t)+1} \leq t + M$.

Let $\mu_M = E[\bar{X}_1]$ we have the Wald's equations:

$$E[\bar{S}_{\bar{N}+1}] = (\bar{m}+1)\mu_M \leq t + M$$

which implies

$$\frac{\bar{m}(t)}{t} + \frac{1}{t} \leq \frac{1}{\mu_M} + \frac{M}{t}$$

and let $t \to \infty$ we have

$$\limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_M}$$

and we know $\lim_{M\to\infty} \mu_M \to \mu$ because the monotonic convergence theorem since $X(M) \uparrow X$. But we also know that $\bar{m}(t) \geq m(t)$ because $\bar{X}_n \leq \bar{X}_n$ and thus

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_M}$$

and this holds for any M. At last we can let $M \to \infty$ and since $\mu_M \to \mu$ we have shown that

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$$

and we proved the claim.

Remark: We cannot derive $m(t)/t \to 1/\mu$ directly from $N(t)/t \to 1/\mu$. Counter example :

$$Y_n(U) = \begin{cases} 0 & \text{if } U > 1/n \\ n & \text{if } U \leq 1/n \end{cases}$$

for $U \sim Uni(0,1)$. Thus $E[Y_n] = 1$ but $Y_n \to 0$ and $E[Y_\infty] = 0$.

Central limit theorem for $N(t)$: i.i.d. interarrival time are F distributed with mean μ and variance σ^2. Then

$$\frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} \to_D \text{Norm}(0,1)$$

proof: Let $r_t = \frac{t}{\mu} + y\sigma\sqrt{t/\mu^3}$,

$$P\left(\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right) = P(N(t) < r_t) = P(S_{r_t} > t)$$
$$= P\left(\frac{S_{r_t} - r_t\mu}{\sigma\sqrt{r_t}} > \frac{t - r_t\mu}{\sigma\sqrt{r_t}}\right)$$

Since $(t - r_t\mu)/\sigma\sqrt{r_t} = -y\sqrt{1 + y\sigma/\sqrt{t\mu}}$ and as $t \to \infty$ the right hand side becomes $-y$. And by CLT, $(S_{r_t} - r_t\mu)/\sqrt{\sigma\sqrt{r_t}}$ converges to Z. As a result we show that as $t \to \infty$, $r_t \to \infty$ and thus,

$$P\left(\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right) = P(Z > -y) = P(Z < y)$$

and this shows that the central limit theorem for $N(t)$.

Remark: Notice $X_n \to_D X$ if and only if $P(X_n \le x) \to P(X \le x)$ for any continuity point x. Need to show the discountinuity point is countable and thus measure zero.

Key Renewal theorems and their applications

$X \ge 0$ and X is lattice if there exists $d \ge 0$ such that $\sum_{n=0}^{\infty} P(X = nd) = 1$.

The largest d that having this property is said to be the period of X.

If $X \sim F$ and X is lattice then F is lattice.

Blackwell's Theorem

1. If F is not lattice then $m(t+a) - m(t) \to a/\mu$ as $t \to \infty$ for all $a \ge 0$.

2. If F is lattice with period d, then E[number of renewal at nd] $\to d/\mu$ as $n \to \infty$.

proof: If $g(a) = \lim_{t\to\infty}[m(t+a) - m(t)]$ exists, then $g(a) = a/\mu$. We can calculate

$$\begin{aligned} g(a+b) &= \lim_{t\to\infty}[m(t+a+b) - m(t)] \\ &= \lim_{t\to\infty}[m(t+a+b) - m(t+a) + m(t+a) - m(t)] \\ &= g(b) + g(a) \end{aligned}$$

which holds for any $a, b \ge 0$ since $g(a), g(b) \ge 0$, we have $g(\cdot)$ is an increasing function and is linear in a. Thus, $g(a) = c \cdot a$ for some constant c.

To show $c = \frac{1}{\mu}$, we define $x_1 = m(1) - m(0)$, $x_2 = m(2) - m(1)$ and $\ldots$ $x_n = m(n) - m(n-1)$. Thus, $g(1) = \lim_{n\to\infty} x_n$. Thus,

$$\lim_n \frac{x_1 + x_2 + \cdots + x_n}{n} = c \quad \Rightarrow \quad \lim_{n\to\infty} \frac{m(n)}{n} = c$$

By elementary renewal theorem, $c = \frac{1}{\mu}$. This shows $g(a) = m(t+a) - m(t) = \frac{a}{\mu}$.

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Now we proof the case when F is lattice.

proof : We assume the period is $d = 1$. Define $U_x = P(\text{there is a renewal at } x) = \sum_{n=0}^{\infty} P(S_n = x)$. want to show $\lim P(\text{renewal at } n) = U_n = \frac{1}{\mu}$.

Goal: If the limit exists and then the limit is $1/\mu$.

If $lim_{n\to\infty} U_n = \alpha$, then

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} U_i = \alpha$$

However,

$$\sum_{i=0}^{m} U_i = \sum_{i=0}^{m} P(\text{renewal at } i) = E\left[\sum_{i=0}^{m} \mathbb{I}(\text{renewal at } i)\right] = E[N(m)]$$

and since $\frac{E[N(m)]}{m} \to \frac{1}{\mu}$, $\alpha = \frac{1}{\mu}$.

This proof above does not prove E[number of renewal at nd] with aribitary d directly. Let N_n denote the number of renewals at nd Then $N(nd) = \sum_{i=1}^{n} N_i$ Thus we have

$$m(nd) = E[N(nd)] = \sum_{i=1}^{n} E[N_i].$$

Now assume $E[N_i]$ exists and it is c then we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{E[N_i]}{d}=\frac{c}{d}=\lim_{n\to\infty}\frac{\sum_{i=1}^{n}E[N_i]}{nd}=\lim_{n\to\infty}\frac{m(nd)}{nd}=\frac{1}{\mu}$$

This shows $E[N_i]\to c=\frac{d}{\mu}$

Key Renewal Theorem If F is not lattice, and h is directly Riemann integrable, then

$$\lim_{t\to\infty}\int_0^t h(t-x)dm(x)=\frac{1}{\mu}\int_0^\infty h(t)dt$$

The Blackwell's theorem is equivalent to Key Renewal theorem. The "$\Leftarrow$" part is using $h(t)=\mathbb{I}(t\in[0,a])$. The "$\Rightarrow$" proof is done by approximating $h(x)$ with step functions.

Directly Riemann Integrable A function $z:\mathbb{R}^+\to\mathbb{R}^+$ is called Riemann integrable in the closed interval $[0,a]$, $a<\infty$ if

$$\lim_{h\downarrow 0}\sum_{k=1}^{a/h}\left(\sup_{(k-1)h\le t<kh}z(t)-\inf_{(k-1)h\le t<kh}z(t)\right)=0.$$

The Riemann integral is defined as

$$\int_0^a z(t)dt=\lim_{h\downarrow 0}h\sum_{k=1}^{a/h}\sup_{(k-1)h\le t<kh}z(t)$$

But how do we define integral on $[0,\infty)$?

A function Z is Riemann integrable on $[0,\infty)$, it if is Riemann integrable on $[0,a]$ for any $a\to 0$, $\lim_{a\to\infty}\int_0^a z(t)dt$ is finite and

$$\int_0^\infty z(t)dt=\lim_{a\to\infty}\int_0^a z(t)dt.$$

Now we can define directly Riemann Integrable:

For $h>0$, $n\in\mathbb{N}$, define $I_n(h)=[(n-1)h,nh)$ and $\mathbb{R}_+=\cup_{n\in\mathbb{N}}I_n(h)$. for function $z:\mathbb{R}^+\to\mathbb{R}^+$ bounded over finite intervals, let

$$\underline{z}_n(h)=\inf\{z(t),t\in I_n(h)\}$$
$$\bar{z}_n(h)=\sup\{z(t),t\in I_n(h)\}$$

Define $\underline{z}_h(t)$ and $\bar{z}_h(t)$ as

$$\underline{z}_h(t)=\sum_{n\in\mathbb{N}}\underline{z}_n(h)\mathbb{I}_{I_n(h)}(t)$$
$$\bar{z}_h(t)=\sum_{n\in\mathbb{N}}\bar{z}_n(h)\mathbb{I}_{I_n(h)}(t)$$

Thus, $\underline{z}_h\le z\le\bar{z}_h$. Then we define two integrals

$$\int_{t\in\mathbb{R}^+}\underline{z}_h(t)dt=h\sum_{n\in\mathbb{N}}\underline{z}_n(h)$$
$$\int_{t\in\mathbb{R}^+}\bar{z}_h(t)dt=h\sum_{n\in\mathbb{N}}\bar{z}_n(h)$$

A function $Z : \mathbb{R}^+ \to \mathbb{R}^+$ is directly Riemann integrable if

$$\lim_{h \downarrow 0} \int_{t \in \mathbb{R}^+} \underline{z}_h(t)dt = \lim_{h \downarrow 0} \int_{t \in \mathbb{R}^+} \bar{z}_h(t)dt$$

and they are finite.
Property:

1. A d.R.I function over $\mathbb{R}^+$ is also Riemann integrable but the converse need not be true.
2. A function is d.R.I if one of the following holds:
   (a) z is monotone non-increasing and lebesgue integrable $\int_0^\infty z(t)dt < \infty$
   (b) z is bounded above by a d.R.I function and is Riemann integrable in $[0, a]$ for $\forall a > 0$.

The key renewal theorem can be very useful. For example if we want to computt the distribution of $S_{N(t)}$, the time of the last renewal prior to time t. For $s < t$:

$$\begin{aligned}
P\{S_{N(t)} \le s\} &= \sum_{n=0}^{\infty} P\{S_n \le s, N(t) = n\} \\
&= \sum_{n=0}^{\infty} P\{S_n \le s, S_{n+1} > t\} \\
&= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^\infty P\{S_n \le s, S_{n+1} > t | S_n = y\} dF_n(y)
\end{aligned}$$

Because $P(S_n \le s | S_n = y) = 0$ when $y > s$, the second integration can be integrated to s. Thus, we have

$$\begin{aligned}
P\{S_{N(t)} \le s\} &= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^s P\{S_n \le s, S_{n+1} > t | S_n = y\} dF_n(y) \\
&= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^s P\{S_{n+1} > t | S_n = y\} dF_n(y) \\
&= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^s P\{X_{n+1} > t - y\} dF_n(y) \\
&= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^s \bar{F}(t - y) dF_n(y)
\end{aligned}$$

Then we can move the sum into the integration and get

$$\begin{aligned}
P\{S_{N(t)} \le s\} &= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^s \bar{F}(t - y) dF_n(y) \\
&= \bar{F}(t) + \int_0^s \bar{F}(t - y) d\left(\sum_{n=1}^{\infty} F_n(y)\right) \\
&= \bar{F}(t) + \int_0^s \bar{F}(t - y) dm(y)
\end{aligned}$$

Here the second integration can be evaluated using the key renewal theorem.

Notice: the distribution of $S_{N(t)}$ consists two parts, one is the discrete part $S_{N(t)} = 0$ and a continuous part which is the integration part.
Also,

$$dF_{S_{N(t)}} = \bar{F}(t-y)dm(y) \quad \text{for } y > 0$$

Sep 23, 2021
Alternating Renewal Process
Starts from time 0, there will be on part and off part for each renewal X_i with on part denoting Z_i and off part denoted by Y_i which is the time for on and off respectively.
Assume $(Z_n, Y_n)_{n=1}^{\infty}$ are iid and there exists dependence between Y_i and Z_i. Want to compute

$$p(t) = P\{\text{System is on at time } t\}$$

with H as the distribution for Z_i and G for Y_i and F for $Z_i + Y_i$.
Theorem If $E[X_n + y_n] < \infty$ and F is non lattice, then

$$\lim_{t\to\infty} p(t) = \frac{E[Z_n]}{E[Z_n] + E[Y_n]}.$$

Proof: Define a renewal as each time the system goes on

$$p(t) = p(\text{on at } t|S_{N(t)} = 0)P(S_{N(t)} = 0) + \int_0^{\infty} P(\text{on at } t|S_{N(t)} = y)dF_{S_{N(t)}}(y)$$

But

$$P(\text{on at } t|S_{N(t)} = 0) = P(Z_1 > t|Z_1 + Y_1 > t) = \frac{\bar{H}(t)}{\bar{F}(t)}$$

For $y < t$,

$$P(\text{ on at } t|S_{N(t)} = y) = p(Z > t-y|Z + Y > t-y) = \frac{\bar{H}(t-y)}{\bar{F}(t-y)}$$

As a result, we have

$$p(t) = \bar{H}(t) + \int_0^t \frac{\bar{H}(t-y)}{\bar{F}(t-y)}\bar{F}(t-y)dm(y) = \bar{H}(t) + \int_0^t \bar{H}(t-y)dm(y)$$

where we used the fact that $dF_{S_{N(t)}} = \bar{F}(t-y)dm(y) \quad \text{for } y > 0$. In the limit $t \to \infty$, $\bar{H}(t) \to 0$. Using Key Renewal theorem, the second part becomes

$$\lim_{t\to\infty} \int_0^t \bar{H}(t-y)dm(y) = \frac{1}{E[X]}\int_0^{\infty} \bar{H}(t)dt = \frac{E[Y]}{E[X] + E[Y]}.$$

We can apply this theorem to the distribution of residual life and age.
Define $Y(t) = S_{N(t)+1} - t$ as the residual life and define $A(t) = t - S_{N(t)}$ as the age at t.
If each renewal is the renew of a bulb. Then the residual life is the time remaining at time t for it to die. The age is the usage time of the bulb at time t.
We want to find $\lim_{t\to\infty} P(A(t) \le x)$ and $\lim_{t\to\infty} P(Y(t) \le x)$ Now let X_n to denote the total age of each bulb. Then we say that if $A(t) < x$ for a given x, it is a on state. Thus Z_n the time for on state will become

$Z_n = \min(X_n, x)$. Because the maximum of the age is x. Then define on state as $Z_n = \min(X_n, x)$ and off state as $Y_n = X - Z_n$. Then

$$\lim P(A(t) \leq x) = \lim P(\text{ on at time t}) = \frac{E[Z]}{E[X]} = \frac{1}{\mu}E[\min(X,x)] = \frac{1}{\mu}\int_0^\infty p(\min(X,x) > y)dy$$
$$= \frac{1}{\mu}\int_0^x \bar{F}(y)dy$$

Similarly, we can find $P(Y(t) \leq x)$. We define the off state to be the last x units of a renewal cycle. Thus, if $Y(t) \leq x$, it means t is within the last x units of the cycle and thus it is in the off state.

$$\lim_{t\to\infty} P(Y(t) \leq x) = \lim_{t\to\infty} P(\text{off at } t) = \frac{E[\min(x,X)}{E[X]} = \frac{1}{\mu}\int_0^x \bar{F}(y)dy$$

Therefore, we find the expected age and life have the same distribution.

We can also compute the mean of $Y(t)$ as

$$E[Y(t)] = E[Y(t)|S_{N(t)} = 0]\bar{F}(t) + \int_0^t E[Y(t)|S_{N(t)} = y]\bar{F}(t-y)dm(y)$$

and

$$E[Y(t)|S_{N(t)} = 0] = E[X_1 - t|X_1 > t] \qquad E[Y(t)|S_{N(t)} = y] = E[X - (t-y)|X > t-y]$$

As a result,

$$E[Y(t)] = E[X-t|X>t]\bar{F}(t) + \int_0^t E[X-(t-y)|X > (t-y)]\bar{F}(t-y)dm(y)$$

Define $h(t) = E[X - t|X > t]\bar{F}(t)$ and we have

$$E[Y(t)] = h(t) + \int_0^t h(t-y)dm(y)$$

We calim that $h(\cdot)$ is d.R.i if $E[X^2] < \infty$ and $h(t) \to 0$ as $t \to \infty$ because

$$h(t) = E[X-t|X>t]\bar{F}(t) = \int_t^\infty (x-t)dF(x) \geq 0$$
$$= -\int_t^\infty (x-t)d\bar{F}(x) = -\int_0^\infty xd\bar{F}(x+t) = -[x\bar{F}(x+t)]_0^\infty - \int_0^\infty \bar{F}(t+x)dx$$
$$= \int_0^\infty \bar{F}(x+t)dx$$

which shows this is a monotonic decreasing function .
Thus

$$E[Y(t)] = h(t) + \int_0^t h(t-y)dm(y) = 0 + \frac{1}{\mu}\int_0^\infty h(t)dt$$

but

$$\int_0^\infty h(t)dt = \int_0^\infty \int_t^\infty (x-t)dF(x)dt = \int_0^\infty \int_0^x (x-t)dtdF(x) = \int_0^\infty \frac{x^2}{2} = \frac{1}{2}E[X^2]$$

As a result, we have

$$\lim_{t\to\infty} E[Y(t)] = \frac{E[X^2]}{2\mu}$$

Note that $S_{N(t)+1} = t + y(t)$ and thus $E[S_{N(t)+1}] = t + E[Y(t)]$ which implies using the Wald equation

$$\mu(m(t)+1) = t + E[Y(t)] \quad \Rightarrow m(t) - \frac{t}{\mu} \to \frac{E[X^2]}{2\mu^2} - 1$$

which is a stronger statement than $m(t)/t \to 1/\mu$.

Regnerate Process

Let $\{X(t), t \geq 0\}$ state space as $\{0, 1, 2, \cdots\}$. The process restarts itself after certain time points. Mathematically, there exist $0 \leq S_1 < S_2 < \cdots$ such that the post S_k process $\{X(S_k + t), t \geq 0\}$ has the same distribution as the original process $X(t)$ and independent of the pre S_k process $\{X(t), 0 \leq t < S_k\}$.

For example, the symmetric random walk process which starts from the origin. So every time the process goes back to the origin, the process starts to repeat itself.

Thus, the time $\{S_1, S_2, \cdots,\}$ constitue the event times of renewal process. We say that a cycle is completed every time a renewal occurs. And the number of cycles is

$$N(t) = \max\{n|S_n \leq t\}$$

and it is embedded renewal process.

Recurrent Markov Chain Theorem: If F the distribution of a cycle, has a density over some interval and if $E(S_1) < \infty$ Then

$$P_j = \lim_{t\to\infty} P(X(t) = j) = \frac{E[\text{amount of time in state j during a cycle}]}{E[\text{time of a cycle}]}$$

Sep 28th, 2021

proof Fix j let $p(t) = P(X(t) = j)$ Then

$$p(t) = P(X(t) = j|S_{N(t)} = 0)\bar{F}(t) + \int_0^t P(X(t) = j|S_{N(t)} = s)\bar{F}(t-s)dm(s)$$

Now we calim that $P(X(t)|S_{N(t)} = s) = P(X(t-s) = j|S_1 > t-s)$. This is because, the process is regeneration, so everything starts again at time s. So time t becomes $t - s$. Also, $S_{N(t)} = s$ means there is no renewal between time t and s and thus $\{S_{N(t)=s}\} = \{S_1 > t - s\}$.

Now we define

$$P(X(t) = j|S_{N(t)} = s) = P(X(t-s) = j|s_1 > t-s) = \tilde{h}(t-s)$$
$$h(t-s) = \tilde{h}(t-s)F(t-s)$$

Notice $h(t) = P(X(t) = j|S_1 > t)P(S_1 > t) = P(X(t) = j, S_1 > t) \leq P(S_1 > t) \to 0$, Thus, we have

$$p(t) = h(t) + \int_0^t h(t-s)dm(s) \to \int_0^\infty h(t)dt\frac{1}{E[S_1]}$$

Now we can define $I(t) = 1$ if $X(t) = j$ and $S_1 > t$ and $I(t) = 0$ otherwise. Thus, $\int_0^\infty I(t)dt$ is the amount of time in the first cycle that $X(t) = j$. As a result,

$$E[\int_0^\infty I(t)dt] = \int_0^\infty E[I(t)]dt = \int_0^\infty p(X(t) = j, S_1 > t)dt = \int_0^\infty h(t)dt$$

which is exactly the number of visits to state j during a cycle.
A minor thing: Is $h(t)$ really d.R.i? Notice $h(t) = P(X(t) = j, S_1 > t) \le P(S_1 > t)$. Since $\int_0^\infty P(S_1 > t)dt = E[S_1] < \infty$ and thus $P(S_1 > t)$ is d.R.I because it is finite and non-increasing. By property 2.(b), we just need to show $h(t)$ is Riemann integrable.

Arcsin law for Symmetry Random Walk
Let Y_1, $Y_2, \cdots$ be iid, and $P(Y_1) = P(Y_1 = -1) = 1/2$. Then the process $Z_n = \sum_{i=1}^n Y_i$ is the random walk process.
Define

$$X_n = \begin{cases} 0 & Z_n = 0 \\ 1 & Z_n > 0 \\ -1 & Z_n < 0 \end{cases}$$

$\{X_n,\ n \ge 0\}$ is a regnerative process that regenerates whenever X_n takes value 0. Then

$$u_n \equiv P\{Z_{2n} = 0\} = \binom{2n}{n} \left(\frac{1}{2}\right)^{2n}$$

And $u_n = \frac{2n-1}{2n} u_{n-1}$ by direct calculation. However

$$P(\text{first visit to 0 in SRW occurs at time } 2n) = \frac{1}{2n-1}\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} = \frac{u_n}{2n-1}$$

by the coin toss and Ballot lead problem (it is like one of the candidates always leads until the $2n$ ballot).
Lemma $P(Z_1 \neq 0, Z_2 \neq 0, \cdots, Z_{2n} \neq 0) = u_n$.
proof By the complements we have

$$\begin{aligned} P(Z_1 \neq 0, Z_2 \neq 0, \cdots, Z_{2n} \neq 0) &= 1 - P\{\text{the SRW hit 0 by time } 2n\} \\ &= 1 - \sum_{k=1}^n P\{\text{SRW hit 0 first time at time} 2k\} \\ &= 1 - \sum_{k=1}^n \frac{u_k}{2k-1} \end{aligned}$$

We want to show that the result equals u_n. We do this by induction. Assume this holds for $n-1$ then

$$\begin{aligned} 1 - \sum_{k=1}^n \frac{u_k}{2k-1} &= 1 - \sum_{k=1}^{n-1} \frac{u_k}{2k-1} - \frac{u_n}{2n-1} \\ &= u_{n-1} - \frac{u_n}{2n-1} = u_{n-1} - \frac{1}{2n-1}\frac{2n-1}{2n}u_{n-1} = u_n \end{aligned}$$

which shows the Lemma.

However, by Sterling $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$, $u_n \sim \frac{1}{\sqrt{n\pi}} \to 0$.
However

$$0 \le P(\text{SRW never returns to 0}) \le P(Z_1 \neq 0, \dots Z_n \neq 0) = u_n \to 0$$

As a result, the probability that it never returns to 0 has probability 0.

Sep 30, 2021
In the SRW problem, we are particular interested in the amount of time of process to be in the positive or negative part. It is not the naive 1/2 as we may have argued from the symmetry prospect.
Proposition For $k = 0, 1, 2, \dots, n$,

$$P\{Z_{2k} = 0, Z_{2k+1} \neq 0, \cdots, Z_{2n} \neq 0\} = u_n u_{n-k}$$

Proof: the probability is

$$\begin{aligned} P\{Z_{2k} = 0, Z_{2k+1} \neq 0, \cdots, Z_{2n} \neq 0\} &= P\{Z_{2k+1} \neq 0, \cdots, z_{2n} \neq 0 | Z_{2k} = 0\} P\{Z_{2k} = 0\} \\ &= u_{n-k} u_k \end{aligned}$$

where the second equality holds because of the regenerate property.
Let $E_{k,n}$ be the event that by time $2n$ the SRW will be positive for $2k$ time units and negative for $2(n-k)$ time units.
Theorem $b_{k,n} = p(E_{k,n}) = u_k u_{n-k}$ with $k = 0, 1, \cdots, n$.
Proof : $n = 1$, $k = 0$, the equality is trivial, $b_{0,1} = b_{1,1} = \frac{1}{2}$ and $u_0 = 1$ and $u_1 = 1/2$.
Assume $b_{k,m} = u_k u_{m-k}$ holds for all $m < n$, $0 \le k \le m$
Case 1 $k = n$, let T be the time of the first time return to 0 then

$$b_{n,n} = \sum_{r=1}^{n} p(E_{n,n} | T = 2r) p(T = 2r) + P(E_{n,n} | T > 2n) P(T > 2n)$$

Given $T = 2r$, the SRW has always been positive in time $(0, 2r)$ with probability 1/2 (either all positive or all negative) Then

$$\begin{aligned} p(E_{n,n} | T = 2r) &= \frac{1}{2} b_{n-r,n-r} \quad \text{regenerate} \\ p(E_{n,n} | T > 2n) &= \frac{1}{2} \\ p(T > 2n) &= p(Z_1 \neq 0, Z_2 \neq 0, \dots, Z_{2n} \neq 0) = u_n \end{aligned}$$

thus, we have

$$\begin{aligned} b_{n,n} &= \frac{1}{2} \sum_{r=1}^{n} b_{n-r,n-r} P(T = 2r) + \frac{1}{2} u_n \\ &= \frac{1}{2} \sum_{r=1}^{n} u_{n-r} P(T = 2r) + \frac{1}{2} u_n \end{aligned}$$

Now focus on the first term:

$$\sum_{r=1}^{n} u_{n-r} P(T=2r) = \sum_{r=1}^{n} P(Z_{2(n-r)}=0)P(T=2r)$$
$$= \sum_{r=1}^{n} P(Z_{2n}=0|T=2r)P(T=2r) \quad \text{by regneration}$$
$$= p(Z_{2n}=0) = u_n$$

This then shows $b_{n,n} = u_n = u_n u_0$ which shows the induction works.
Case 2 $0 < k < n$ we have

$$b_{k,n} = \sum_{r=1}^{n} p(E_{k,n}|T=2r)p(T=2r)$$

where we do now worry about the case for $T > 2n$ in which case, the first touch is beyond $2n$ and $E_{k,n}$ with $k > 0$, $n-k > 0$ which requires some time positive and some time negative would have zero probability. Given $T = 2r$, the SRW has been always positive or always negative in $(0, 2r)$ withe equal probability. If it stays in positive before $T = 2r$, need to spend $2k - 2r$ time in positive after time $2r$. If it stays in negative before $T = 2r$, need to spend total $2k$ time afterwards. Thus,

$$b_{k,n} = \frac{1}{2}\sum_{r=1]}^{k} b_{k-r,n-r} P(T=2r) + \frac{1}{2}\sum_{r=1}^{n-k} P(T=2r)$$
$$= \frac{1}{2} u_{n-k} \sum_{r=1}^{k} u_{k-r} P(T=2r) + \frac{1}{2} u_k \sum_{r=1}^{n-k} u_{n-r-k} P(T=2r) \quad \text{induction}$$

Now the first term:

$$\sum_{r=1}^{k} u_{k-r} P(T=2r) = \sum_{r=1}^{k} p(Z_{2k-2r}=0)P(T=2r)$$
$$= \sum_{r=1}^{k} p(Z_{2k=0}|T=2r)p(T=2r) \quad \text{regenerate}$$
$$= p(Z_{2k}=0) = u_k$$

similarly, the second term is $u_k u_{n-k}$ and this completes the proof.
Let X be the amount of time units the SRW stay positive by time $2n$. Then by the above theorem:

$$p(X=2K) = p(E_{k,n}) = u_k u_{n-k}$$

For large k, n, by Sterling,

$$u_k u_{n-k} \sim \frac{1}{\pi\sqrt{k(n-k)}}$$

Thus for $x \in (0,1)$, we have

$$P(X \le 2nx) = p(\frac{X}{2n} \le x) = \sum_{k=0}^{nx} u_k u_{n-k} \sim \frac{1}{\pi}\int_0^{nx} \frac{1}{\sqrt{y(n-y)}}dy = \frac{1}{\pi}\arcsin(\sqrt{x})$$

Thus, $X/2n$ converges in distribution to an arcsin law distribution random variable A with $p(A \leq x) = \frac{2}{\pi}\arcsin(\sqrt{x})$

There is a proposition:

Proposition For a regenerate process with $E[S_1] < \infty$ Then with probability 1

$$\lim_{t\to\infty} \frac{\text{Amount of time in j during } [0,t]}{t} = \frac{E[\text{time in j during a cycle}]}{E[\text{time of a cycle}]}$$

However, if we define $\tilde{X}_n = \text{sign}(Z_n)$ with $\tilde{X}_n = 0$ iff $Z_n = 0$. Then

$$\frac{\text{amount of time } X_n = 1}{2n} \to \frac{X_n}{2n} \sim A \neq \frac{1}{2}$$

but

$$\frac{E[\tilde{X} = 1 \text{ in one cycle}]}{E[T]} = p(X_n = 1) = P(Z_n > 0) = \frac{1}{2}$$

this means the proposition does not work and this means the expectation of the cycle length is infinity.

Oct 5, 2021

Markov Chain (Discrete Time)

The Markov property: $P(X_{n+1} = j | X_n = i, X_{n-1}) = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j | X_n = j) = p_{ij}$

A more general version is $P(X_{n+1} = j X_{n_k} = i, \cdots) = P(X_{n+1} = j | X_{n_k} = i)$ which means only the most recent observation matters.

Example: A random walk on a graph with (V, E). The state space is $\{$all vertices$\}$. The transition probability $P_{v_i,v_j} = \frac{1}{d(v_i)}$ when there exists an edege that connects v_i and v_j.

Single server queue and customer arrives according to Poisson. The number of customers in the system at t, denoted at $X(t)$. $X(t)$ is not a Markov when the service time is memoriless.

Denote X_n the number of customers left behind by the n-th departure and Y_n the number of customers during the service period of the $n+1$ customer. Then X_n is Markov. Clearly,

$$X_{n+1} = \begin{cases} X_n - 1 + Y_n & X_n > 0 \\ Y_n & X_n = 0 \end{cases}$$

and the distribution of Y_n is

$$\begin{aligned} P(Y_n = j) &= \int P(Y_n = j | \text{service time for n+1 customer is x}) dG(x) \\ &= \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dG(x) \end{aligned}$$

Thus, $P_{ij} = P(X_{n+1} = j | X_n = i) = P(Y_n = j - i + 1)$ and since Y_n is independent of X_n, the process is Markovian.

Chapman-Kolmogorov Equation

p_{ij}^n is the probability that a process moves from i to j in n steps.

$$p_{ij}^n = p(X_{n+m} = j | X_m = i)$$

The Chapman-Kolmogorov Equation says

$$p_{ij}^{n+m} = \sum_{k \in \mathcal{S}}^{\infty} P_{ik}^{n} P_{kj}^{m}$$

The proof is quite simple:

$$\begin{aligned} P - ij^{m+n} = P(X_{n+m} = j | X_1 = i) &= \sum_k P(X_{n+m} = j, X_m = k | X_1 = i) \\ &= \sum_k P(X_{n+m} | X_m = k, X_1 = j) P(X_m = k | X_1 = j) \end{aligned}$$

By Markovian, we have shown the result.
Using this result, we can show the n step transition matrix $p^{(n)}$ is just p^n.
The state j is said to be accessible from state i if for some $n \geq 0$, $p_{ij}^n > 0$. If i and j are accessible to each other, then we say the two states communicate $i \leftrightarrow j$.
Proposition: communicate is an equivalence relation. Partition the states using the communicate equivalence relation.
Markov Chain is **irreducible** if there is only one communicate class.
A state i is said to have period d if $P_{ii}^n = 0$ whenever n is not divisible by d. d is the greatest integer having this property.
A state with period 1 is said to be aperiodic.
Define $d(i)$ as the period of state i. Then if $i \leftrightarrow j$, then $d(i) = d(j)$.
Proof: suppose $p_{ii}^s > 0$, then $P_{jj}^{n+m} \geq P_{ji}^n P_{ij}^m > 0$ and $P_{jj}^{n+m+s} \geq P_{ji}^n P_{ii}^s P_{ij}^m > 0$. By definition, $(n+m)/d(j) \in \mathbb{N}$. Then $(n+m+s)/d(j) \in \mathbb{N}$. Thus, $S/d(j) \in \mathbb{N}$ for $\forall s, P_i i^s > 0$. Thus, $d(i)$ divides $d(j0$. Reverse the argument, we see $d(i) = d(j)$.
Define $f_{ij}^n = P$(starting in i, the first transition onto j occurs at time n). Thus,

$$f_{ij}^n = p(X_n = j, X_k \neq j, k = 1, 2, \ldots, n-1 | X_0 = i).$$

Then the quantity

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n = p\{\text{prob of ever making a transition to j from i}\}$$

A state j is recurrent if $f_{jj} = 1$ and transient if $f_{jj} < 1$.
Proposition If state j is recurrent, if and only if $\sum_{n=1}^{\infty} p_{jj}^n = \infty$.
proof: j is recurrent, then with probability 1, a process starting at j will eventually return after a finite number of steps. Thus, with probability 1, the number of visits to j is infinity. Then

$$E[\text{number of visits to j} X_0 = j] = \infty$$

Define I_n as the indicator that $X_n = j$. Thus,

$$E[\text{number of visits to j} | X_0 = j] = \sum_n E[I_n | X_0 = j) = \sum_n p_{jj}^n = \infty$$

But if the state is transient, then the number of visits is finite. Thus, $\sum_{n=1}^{\infty} p_{jj}^n < \infty$. Because a state is either recurrent or transient. we showed the proposition.

Oct 7th, 2021

Corollary: If i is recurrent and $i \leftrightarrow j$, then j is recurrent.

proof $\exists m, n$ $P_{ij}^n > 0$ and $P_{ji}^m > 0$ by the definition. Then for $\forall s \geq 0$,

$$P_j j^{m+n+s} \geq P_{ji}^m P_{ii}^s P_{ij}^n$$

Thus,

$$\sum_s P_{jj}^{m+n+s} \geq P_{ji}^m \sum_s P_{ii}^s P_{ij}^n = \infty$$

Thus, j is recurrent.

Example: Let's consider the simple random walk. Because all states communicate, they either be all recurrent or all transient by the above corollary. We thus only focus on the state 0. We know that $P_{00}^{2n+1} = 0$ since odd number cannot make the walk back to origin. However,

$$P_{00}^{2n} = \binom{2n}{n} p^n (1-p)^n \sim \frac{[4p(1-p)]^n}{\sqrt{\pi n}}$$

Since $4p(1-p) < 1$ when $p \neq 1/2$ and $4p(1-p) = 1$ when $p = 1/2$. Thus,

$$\sum_{n=1}^{\infty} p_{00}^{2n} \begin{cases} = \infty \text{ when } p = \dfrac{1}{2} \text{ and thus recurrent} \\ < \infty \text{ when } p \neq \dfrac{1}{2} \text{ and thus transient} \end{cases}$$

Random Walk on $\mathbb{Z}^d$

Now if we consider the random walk in $\mathbb{Z}^d$. Now the walk possibility is $e_j = (0, 0 \ldots, 1, \ldots, 0)$ for j denoting the direction $1, 2, \ldots, d$. The proability would be

$$p(e_j) = p(-e_j) = \frac{1}{2d}$$

In this case, we still need even number of walks to back to the origin. This is because for each dimension to be zero, we need even number of walks in each of the dimension.

Local CLT

Consider

$$\bar{p}(x) = \left(\frac{d}{2\pi n}\right)^{d/2} e^{-\frac{|x|^2 d^2}{2n}}$$

Then $\exists c > 0$, such that $\forall n x = (x_1, \ldots, x_d)^T$,

$$|P(S_n = x) - \bar{p}_n(x)| < \frac{c}{n^{(d+2)/2}}$$

Now we let $x = 0$, $\bar{p}_n = \frac{C_d}{n^{d/2}}$ and the error term takes higher power of n in the denominator and is less important when $n \to \infty$. Thus, $p(S_n = 0) \sim n^{-d/2}$. Thus, the random walk is recurrent when $d = 2$ and transient when $d \geq 3$.

Let $X_1, X_2, \dots, X_n \sim (0, \Sigma)$. with support on $\mathbb{Z}^d$. The general Local CLT says for

$$\bar{p}_n(x) = \frac{1}{(2\pi n)^{d/2}\sqrt{\det\{\Sigma\}}} e^{-\frac{1}{2n}x\Sigma^{-1}x}$$

we have $\exists c > 0$,

$$|p_n(x) - \bar{p}_n(x)| < \frac{c}{n^{(d+2)/2}} \quad \forall n, x$$

Corrollary If $i \leftrightarrow j$ and j is recurrent, then $f_{ij} = 1$.
proof suppose $X_0 = i$, $\exists n$, $P_{ij}^n > 0$. If $X_n \neq j$, we miss opportunity 1. Let T_1 be the next time we enter i and this number is finit since i is recurrent. $X_{T_1} = i$, $X_{T_1+n} = j$ with probability $p_{ij}^n > 0$. If we miss again, we wait another time T_2 to back to state i. Thus, the number of miss before the first success is $1/P_{ij}^n$. Thus, the number of trials to get a first success will have zero probability to be infinity since the mean of the number of trials is finite.
Let $N_j(t)$ denote the number of transition into j before t.
If $X_0 = j$, Then $N_j(t)$ will be a renewal process with interarrival distribution $\{f_{jj}^n\}_{n=1}^\infty$.
If $X_0 = i$, then $N_j(t)$ will be a delayed renewal process.
Delayed Renewal Process
$X_1 \sim G$, $X_2, X_3, \dots \sim F$ and X_i are independent. $S_0 = 0$ and $S_n = \sum X_i$ and

$$N_D(t) = \sup\{n | S_n \leq t\}$$

is the delayed renewal process. Let $m_D(t) = E[N_D(t)]$ and $\mu = \int x dF(x)$, the mean of F.
Proposition:

(i) with probability 1, $\frac{N_D(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$

(ii) $\frac{m_D(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$

(iii) If F is not lattice, then $m_D(t+a) - m_D(t) \to \frac{a}{\mu}$ as $t \to \infty$

(iv) If F and G are lattice with period d, $\mathbb{E}$[number of renewal at nd] $\to d/\mu$ as $n \to \infty$

If j is transient, then $\sum_{n=1}^\infty p_{ij}^n < \infty$ for all i, implying that $p_{ij}^n \to 0$.
Let μ_{jj} equal the expected number of transitions needed to return to state j. Then $\mu_{jj} = \sum_{n=1}^\infty n f_{jj}^n = \infty$ if j is transient.
Limit Theorem If $i \leftrightarrow j$, then

(i) $p\left[\lim \frac{N_j(t)}{t} = \frac{1}{\mu_{jj}} | X_0 = i\right] = 1$

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n p_{ij}^k = \lim_{n\to\infty} \frac{E[N_j(n)]}{n} = \frac{1}{\mu_{jj}}$

(iii) If j is aperiodic, then $\lim_{n\to\infty} p_{ij}^n = \frac{1}{\mu_{jj}}$

(iv) If j has period d, then $\lim_{n\to\infty} p_{ij}^{nd} = \frac{d}{\mu_{jj}}$

If State j is recurrent. Then

- $\mu_{jj} < \infty \quad \Leftrightarrow \quad$ j is a positive recurrent state
- $\mu_{jj} = \infty \quad \Leftrightarrow \quad$ j is a null recurrent state

proposition The positive null recurrence is a class property. That is if $i \leftrightarrow j$, j is positive recurrent, then i is also positive recurrent.
A positive recurrent, aperiodic state is called ergodic.

Oct 12th, 2021
Define a probability distribution $\{p_j, j \geq 0\}$ is said to be stationary for the MC if

$$p_j = \sum_{i=0}^{\infty} p_i p_{ij}, \quad j \geq 0$$

If $X_n \sim \{p_j\}_{j\geq 0}$ (stationary distribution), then $X_i \sim \{p_j\}$ for $i \geq 1$. As a result, $\{X_n, X_{n+1}, \ldots, X_{n+m}\}$ for $\forall m \geq 0$, has the same joint distribution for each n. This is because

$$P(X_n, X_{n+1}, X_{n+2}) = P(X_{n+2}|X_{n+1}, X_n)P(X_{n+1}|X_n)P(X_n) = P(X_{n+2}|X_{n+1})P(X_{n+1}|X_n)P(X_n)$$

because $P(X_n)$ is the stationary distribution and $P(X_{n+2}|X_{n+1})$ only depends on p_{ij} matrix, the above equation right hand side does not depend on n.
Theorem An irreducible aperiodic MC belongs to one of the following two cases

1. Either the states are all transient or all null recurrent; in this case, $p_{ij}^n \to 0$ as $n \to \infty$ for all i, j and there exists no stationary distribution.
2. All states are positive recurrent, that is

$$\pi_j = \lim_{n\to\infty} p_{ij}^n > 0 = \frac{1}{\mu_{jj}}$$

   In this case $\{\pi_j, j = 0, 1, 2, \ldots\}$ is a stationary distribution and is unique. (Ergodic MC)

Proof for (2). For all finite number M, $\sum_{j=1}^{M} \pi_j \leq 1$ and thus, $\sum_{j=1}^{\infty} \pi_j \leq 1$. Now

$$p_{ij}^{n+1} = \sum_{k=0}^{\infty} p_{ik}^n p_{kj} \geq \sum_{k=0}^{M} p_{ik}^n p_{kj}$$

Let $n \to \infty$

$$\pi_j \geq \sum_{k=0}^{M} \pi_k p_{kj} \quad \forall M \quad \Rightarrow \pi_j \geq \sum_{k=0}^{\infty} \pi_k p_{kj}$$

Suppose the inequality is strict for states j,

$$\sum_{j}^{\infty} \pi_j > \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \pi_k p_{kj} = \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{\infty} p_{kj} = \sum_{k=0}^{\infty} \pi_k$$

Thus, only equality can hold. Thus, π_i is a stationary distribution. We should also demonstrate the uniqueness. Now consider a stationary distribution p_i

$$p_j = p(X_n = j) = \sum_{i=0}^{\infty} p(X_n = j | X_0 = i) p(X_0 = i) = \sum_{i=0}^{\infty} p_{ij}^n p_i$$

Thus, $p_j \geq \sum_{i=0}^{M} p_{ij}^n p_i$ for all M thus, let $M \to \infty$ and $n \to \infty$, we get

$$p_j \geq \sum_{i=0}^{\infty} \pi_j p_i = \pi_j$$

But also, since $p_{ij}^n \leq 1$

$$p_j \leq \sum_{i=0}^{M} p_{ij}^n p_i + \sum_{i=M+1}^{\infty} p_i$$

Let $n \to \infty$, we have

$$p_j \leq \sum_{i=0}^{M} \pi_j p_j + \sum_{i=M+1}^{\infty} p_i \quad \Rightarrow p_j \leq \sum_{i=0}^{\infty} \pi_j p_i = \pi_j$$

As a result, we showed $p_i = \pi_i$.

Theorem An irreducible MC X_n on a finite state space $\mathcal{X}$ has a unique stationary distribution. Furthermore, if MC is not only irreducible but also aperiodic, then for any initial distribution v,

$$\lim_{n \to \infty} p^v(X_n = j) = \pi(j), \quad \forall j \in \mathcal{X}$$

Example M/G/1 queue problem. X_n is the number of customers left behind by the n-th departure. define $a_j = p(j$ arrivals during a service period)

$$a_j = \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^j}{j!} dG(x)$$

X_n is MC with $p_{0j} = a_j$ $p_{ij} = a_{j-i+1}$ if $i > 0$ and $j \geq i - 1$, and $p_{ij} = 0$ otherwise. This problem depends on the parameter $\rho = \sum_{j=0}^{\infty} j a_j$, the average number of arrivals during a service period. We can rewrite ρ as

$$\rho = \mathbb{E}[\text{number of arrivals in service time}] = \mathbb{E}[\mathbb{E}[\text{arrival in time S}|S]] = \mathbb{E}[\lambda S] = \lambda E[S]$$

If $\rho < 1$ this is positive recurrent and there exists a stationary distribution.

In order to solve for the stationary distribution, we need to solve $\pi = \pi P$. Specifically, the equation is

$$\pi_j = \pi_0 a_j + \sum_{i=1}^{j+1} \pi_i a_{j-i+1}, \quad j \geq 0.$$

We can use the generating function approach to solve it.

$$\pi(s) = \sum_{j=0}^{\infty} \pi_j s^j \quad A(s) = \sum_{j=0}^{\infty} a_j s^j$$

Thus, the generating function becomes

$$\pi(s) = \pi_0 A(s) + \sum_{j=0}^{\infty} \sum_{i=1}^{j+1} \pi_i a_{j-i+1} s^j = \pi_0 A(s) + \frac{1}{s} \sum_{i=1}^{\infty} \pi_i s^i \sum_{j=i-1}^{\infty} a_{j-i+1} s^{j-i+1}$$

Note the sum of $\pi_i s^i$ is from $i = 1$ and thus, we have

$$\pi(s) = \pi_0 A(s) + (\pi(s) - \pi_0)A(s)/s \quad \Rightarrow \pi(s) = \frac{(s-1)\pi_0 A(s)}{s - A(s)}$$

We then take the limit $s \to \infty$ to get the sum of π_j.

$$\lim_{s\to 1} \pi(s) = \pi_0 \lim \frac{s-1}{s - A(s)} = \pi_0 \frac{1}{1 - A'(1)} = \pi_0 \frac{1}{1-\rho} = \sum_{j=0}^{\infty} \pi_j = 1$$

when $\rho < 1$, we can find $\pi_0 = (1 - \rho)$.

Oct 21st, 2021

Gambler's Ruin Problem

The win of a gamble has probability as p. Define the probaility that the gambler's fortune reaches N before 0 as $f_{i,N}$ when his original fortune is i. Thus,

$$\alpha_i = f_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ \dfrac{i}{N} & \text{if } p = 1/2 \end{cases}$$

Let B be the number of bets until the gambler's reaches either 0 or N. Thus,

$$B = \min \left\{ m | \sum_{j=1}^{m} x_j = -i \text{ or } \sum_{i=1}^{m} X_j = N - i \right\}$$

Thus, B is a stopping time. Then the sum

$$\sum_{j=1}^{B} X_j = \begin{cases} -i & \text{w.p. } 1-\alpha_i \\ N-i & \text{w.p. } \alpha_i \end{cases}$$

By Wald's equation

$$E\left[\sum_{j=1}^{B} X_j\right] = (1-\alpha_i)(-i) + \alpha_i(N-i) = E[X_i]E[B] = (2p-1)E[B]$$

Then we can derive the expectation of B.

In Finite state Markov Chain, denote $T = \{1, 2, \ldots, t\}$ denotes the set of transient states and write

$$Q = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1t} \\ p_{21} & p_{22} & \cdots & p_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ p_{t1} & p_{t2} & \cdots & p_{tt} \end{pmatrix}$$

For $i, j \in T$, m_{ij} be the expected number of time periods spent in state j given that the chain starts in i.

$$\begin{aligned} m_{ij} &= \delta(i,j) + \sum_k P_{ik} m_{kj} \\ &= \delta(i,j) + \sum_{k=1}^{t} p_{ik} m_{kj} + \sum_{k \notin T} p_{ik} m_{kj} \end{aligned}$$

But, $m_{kj} = 0$ since recurrent class cannot go out to transient state j. Thus, we just need to solve

$$m_{ij} = \delta(i,j) + \sum_{k=1}^{t} P_{ik} m_{kj}$$

Thus, we are solving $M = I + QM$. Thus, $M = (I - Q)^{-1}$
Another way of calculating this is noticing that

$$\begin{aligned} m_{ij} &= E[\text{num of transitions into states } |X_0 = i] \\ &= E\left[\sum_{n=1}^{\infty} I_n(j)|X_0 = j\right] \\ &= \sum_{t=1}^{\infty} E\left[\sum_{n=1}^{\infty} I_n(j)|X_0 = i, N = t\right] P(N = t|X_0 = i) \\ &= \sum_{t=1}^{\infty} m_{jj} f_{ij}^t = m_{jj} f_{ij} \end{aligned}$$

Thus, $m_{ij} = m_{jj} f_{ij}$.

Branching Process
X_0 is the size of the zeroth generation. X_1 the number of all off spring of the zeroth generation. X_n is the size of the n-th generation. $\{X_n\}$ is a branching process. Thus,

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$

is a Markov process, where Z_i represents the number of offspring of the i-th individual in the $n-1$ generation. We are interested in the probability that the population ever die out starting with a single individual, π_0. Thus, we have

$$\pi_0 = \sum_{j=0}^{\infty} P(\text{die out}|X_1 = j) P_j = \sum_{j=0}^{\infty} \pi_0^j P_j.$$

Theorem. Suppose that $P_0 > 0$ and $P_0 + P_1 < 1$, then

(i) π_0 is the smallest positive number satisfying $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j p_j$

(ii) $\pi_0 = 1$ if $\mu \leq 1$.

proof Since $\pi \geq 0$ satisfy the equation. We can show that $\pi \geq p(X_n = 0)$. For $\pi = \sum_{j=0}^{\infty} \pi^j p_j \geq \pi^0 p_0 = p(X_1 = 0)$.
Assume $\pi \geq p(X_n = 0)$, Then

$$P(X_{n+1} = 0) = \sum_{j=0}^{\infty} p(X_{n+1} = 0|X_1 = j) p_j = \sum_{j=0}^{\infty} p(X_n = 0)^j p_j \leq \sum_{j=0}^{\infty} \pi^j p_j = \pi$$

by induction and we showed the claim that $\pi \geq p(X_n = 0)$.

We also know that

$$\{\text{population die out}\} = \cup_{n=1}^{\infty}\{X_n = 0\}$$

and since $\{X_n = 0\} \subset \{X_{n+1} = 0\}$. Thus,

$$\pi \geq \lim_{n\to\infty} p(X_n = 0) = P(\cup_{n=1}^{\infty}\{X_n = 0\}) = p(\text{die out})$$

This shows $p(\text{population die out}) = \pi_0$. This shows π_0 is the smallest solution to the equation.
Consider Generating function $\pi(s) = \sum_{j=0}^{\infty} s^j p_j$. Assume $p_0 + p_1 < 1$. The second derivative of $\phi(s)$ is

$$\phi''(s) = \sum_{j=2}^{\infty} j(j-1)s^{j-2}p_j > 0$$

Thus, $\phi(s)$ is strictly convex in the interval $s \in (0,1)$. However, we know $\phi(0) = p_0$ and $\phi(1) = 1$. Since $p_0 > 0$, the function either is always above $y = x$ line or cross it by the convex assumption.
This shows $\phi(s) > s$ or $\phi(s) = s$. In the first case, the function is always above $y = x$ and it is convex, thus, the derivative of $\phi(s)$ at $s = 1$ must be less than 1. In the second case, $\phi'(s) > 1$ at $s = 1$. In the first case, $\pi_0 = 1$ and in the second case, $\pi_0 < 1$. But the first derivative of $\phi(s)$ is μ. Thus, we showed when $\mu < 1$ $\pi = 1$ and when $\mu > 1$, $\pi < 1$.

Oct 26th, 2021
Continuous Time Markov Chain
Assume state space is countable and we have Markov property

$$P\{X(t+s) = j|X(s) = i, X(u) = x(u), 0 \leq u < s\} = P\{X(t+s) = j|X(s) = i\}$$

if the conditional probability is independent of s then the MC is said to have stationary and sometimes called homogeneous transition probability.
Because we only consider discrete state space, the MC would just jump through time. (If we plot state vs time, we would see discontinuity at the time of jumping).
Denote τ_i as the amount of time that the process stays in state i before making a transition into a different state. It it interesting to consider $P(\tau_i > s+t|\tau_i > s)$ and we would find that $P(\tau_i > s+t|\tau_i > s) = P(\tau_i > t)$ according to Markov property. Basically,

$$P(\tau_i > s+t|X(s) = i, \dots) = P(\tau_i > s+t|X(s) = i) = P(\tau_i > t)$$

Thus, according to memoryless, we find the distribution for τ_i must be exponential.
When the process leaves state i, it will next enter state j with probability P_{ij} when $\sum_{j\neq i} P_{ij} = 1$.
Let v_i denote the inverse exponential mean for state i. It denotes the inverse of the waiting time to jump.

(i) $v_i = \infty$, the state i is called instantaneous state

(ii) $v_i = 0$, the state is called absorbing state.

we would assume $0 \le v_i < \infty$.
The amount of time the process spends in state i and the next state visited must be **independent**. Otherwise, we could use the length of the stay to infer where the state would go next and it is clearly not memoryless.
A continuous time MC is **regular** if with probability 1 the numb er of transitions in any finite length of time is finite. We can easily construct a non-regular MC by that $P_{i,i+1} = 1$ and $v_i = i^2$. Then $\tau_i \sim \text{Exp}(v_i^2)$. Thus, the expected time of state would be

$$E\left[\sum_{i=1}^{\infty} \tau_i\right] = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

As a result, $P(\sum_{i=1}^{\infty} \tau_i < t) > 0$. Thus, there must be infinite jumps during the finite time.
Define $q_{ij} = v_i P_{ij}$ and it describes the rate when in state i that the process makes a transition to state j. We call it the **transition rate** from state i to j. Also define

$$P_{ij}(t) = \ P(X(t+s) = j | X(s) = i)$$

as the transition probability from state i to j in time t.
Example: Birth and Death process. The birth rate as $\lambda_i = q_{i,i+1}$ and death rate $\mu_i = q_{i,i-1}$. We can find the jump rate

$$v_i = \sum_{j \neq i} v_i P_{ij} = \sum_{j \neq i} q_{ij} = \lambda_i + \mu_i$$

v_i is number of jumps in one unit of time and it must be equal to the total number of new commers to other states in the unit of time. We can also find

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}.$$

A Birth Death process is said to be **pure birth** process if $\mu_i = 0$ for all i. Example: Poisson process and **Yule process** : Each member acts independently and give birth at an exponential rate λ.

M/M/S Queue Arrival is Poisson process with λ and service time is Exponential μ and S is the number of the servers. $X(t)$ is the number of customers in the system at time t.
This is a brith death processs with birth and death rates as

$$\lambda_n = \lambda \quad \mu_n = \begin{cases} n\mu & \text{if } 1 \le n \le s \\ s\mu & \text{if } n > s \end{cases}$$

where we used the fact that the minimum of n independent Exponential λ has rate $n\lambda$.

Yule Process with $X_0 = 1$ Denote T_i is the time between $(i-1)$-th brith and i-th birth. Since there are i people, the birth rate will be $i\lambda$ and thus $T_i \sim Exp(i\lambda)$. We are specifically interested in $P_{ij}(t) = p(X(t) = j | X(0) = i)$. We first need to find the distribution for $T_1 + \cdots + T_j$.

$$\begin{aligned} &P(T_1 \le t) = 1 - e^{-\lambda t} \\ &P(T_1 + T_2 \le t) = \int_0^t P(T_1 + T_2 \le t | T_1 = x) \lambda e^{-\lambda x} dx = \left(1 - e^{-\lambda t}\right)^2 \end{aligned}$$

By induction, we can find that

$$P(T_1 + T_2 + \cdots + T_j \le t) = \left(1 - e^{-\lambda t}\right)^j$$

But we know that $\{T_1 + \cdots + T_j \le t\} = \{X(t) \ge j+1\}$ and thus we find that

$$P(X(t) \ge j+1) = P(T_1 + \cdots + T_j \le t) = \left(1 - e^{-\lambda t}\right)^j$$

which implies that

$$P_{ij}(t) = P(X(t) \ge j) - P(X(t) \ge j+1) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{j-1}$$

which looks like a geometric distribution with $p = e^{-\lambda t}$.
Now if Yule process stars at $X(0) = i$, the size at time t would be distributed as $\sum_{h=1}^{i} G_h$ with $G_h \sim$ geometric($e^{-\lambda t}$). This is because each initial person in X_0 at time t will give offspring distributed as geometric($e^{-\lambda t}$) and then we just add all initial persons together. The sum of geometric distribution would be

$$P_{ij}(t) = \binom{j-1}{i-1} e^{-\lambda t i}\left(1 - e^{-\lambda t}\right)^{j-i}$$

Proposition Consider Yule process with $X(0) = 1$ and given that $X(n) = n+1$, the birth times $S_1, \ldots, S_n$ are distributed as the ordered values from a sample of size n from a population having density

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda(t-x)}}{1 - e^{-\lambda t}} & 0 \le x \le t \\ 0 & \text{otherwise} \end{cases}$$

proof
$S_i = T_1 + \cdots + T_i$ is the i-th birth time. for $0 < s_1 < s_2 < \cdots < s_n < t$, we have

$$\begin{aligned} &P(S_1 = s_1, S_2 = s_2, \ldots, S_n = s_n | X(t) = n+1) \\ &= \frac{P(T_1 = s_1, T_2 = s_2 - s_1, \ldots, T_n = s_n - s_{n-1}, T_{n+1} > t - s_n)}{P(X(t) = n+1)} \\ &= \frac{\lambda e^{-\lambda s_1} 2\lambda e^{-2\lambda(s_2 - s_1)} \ldots n\lambda e^{-n\lambda(s_n - s_{n-1})} e^{-(n+1)\lambda(t-s_n)}}{P(X(t) = n+1)} \\ &= Ce^{-\lambda(t-s_1)} e^{-\lambda(t-s_2)} \ldots e^{-\lambda(t-s_n)} \end{aligned}$$

where C does not depend on $s_1, s_2, \ldots, s_n$ and we know the order statsitics is of the form

$$f(s_1, \ldots, s_n | X(t) = n+1) = n! \prod_{i=1}^{n} f(s_i)$$

which would give us $f(s_i) \sim e^{-\lambda(t-s_i)}$ and we have the above result.

Oct 28th, 2021
Example: We would also like to study the expected age at t.

$$A(t) = a_0 + t + \sum_{i=1}^{X(t)-1} (t - S_i)$$

where a_0 is the age at time $t = 0$ of the initial individual. Thus,

$$E[A(t)] = E[A(t)|X(t) = n+1] = a_0 + t + E[\sum_{i=1}^{n}(t - S_i)|X(t) = n+1]$$

Because given $X(t) = n+1$, S_i are ordered statistics of the above distribution $f(x)$ we have

$$E[A(t)|X(t) = n+1] = a_0 + t + n\int_0^t (t-x)f(x)dx$$
$$\Rightarrow \quad E[A(t)] = a_0 + t + (E[X(t)] - 1)\int_0^t (t-x)f(x)dx = a_0 + \frac{e^{\lambda t} - 1}{\lambda}$$

An alternative way is by noticing that

$$A(t) = a_0 + \int_0^t X(s)ds = a_0 + \frac{e^{\lambda t} - 1}{\lambda}$$

Example: A single epidemic model (SIR, SIRD model)

Suppose we have m individual at time 0, and one infected and $m-1$ susceptible. We need some assumption:

(1) Once infected, remain infected forever

(2) At any time interval h, any given infected person will cause with probability $\alpha h + o(h)$ any given susceptible to become infected

$X(t)$ is the number of infected individuals in the population. Pure birth process

$$\lambda_n = \begin{cases} (m-n)n\alpha & \text{for } n = 1, \ldots, m-1 \\ 0. & \text{for } n = m \end{cases}$$

Let T_i denote the time to go from i infected people to $(i+1)$ infected people. We have interested in $\mathbb{E}[T]$ for $T = \sum_{i=1}^{m-1} T_i$. Thus, $T_i \sim \text{Exp}(\lambda_i)$ with $\lambda_i = (m-i)i\alpha$. Thus, the expected time would be

$$\mathbb{E} = \sum_{i=1}^{m-1}\mathbb{E}(T_i) = \frac{1}{\alpha}\sum_{i=1}^{m-1}\frac{1}{i(m-i)} \quad \text{Var}[T] = \frac{1}{\alpha^2}\sum_{i=1}^{m-1}\frac{1}{i^2(m-i)^2}$$

We can do the sum as

$$E[T] = \frac{1}{m\alpha}\sum_{i=1}^{m-1}\left(\frac{1}{m-i} + \frac{1}{i}\right) \approx \frac{1}{m\alpha}\int_1^{m-1}\left(\frac{1}{m-t} + \frac{1}{t}\right)dt = 2\frac{\log(m-1)}{m\alpha}$$

Kolmogorov Differential Equation

We are interested in the equation for $P_{ij}(t) = p(X(t+s) = j|X(s) = i)$ We first have a Lemma:

Lemma

(i) $\lim_{t\to 0}\frac{1 - P_{ii}(t)}{t} = v_i$

(ii) $\lim_{t\to 0}\frac{P_{ij}(t)}{t} = q_{ij}$

and another Lemma:

Lemma : For s, t, $P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{jk}(t)P_{kj}(s)$.

From the equation we can write

$$P_{ij}(t+h) = \sum_{k} P_{ik}(h)P_{kj}(t) \quad \Rightarrow P_{ij}(t+h) - P_{ij}(t) = \sum_{k \neq i} P_{ik}(h)P_{kj}(t) - (1 - P_{ii}(h))P_{ij}(t)$$

we can divide h and get

$$\lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h \to 0} \sum_{k \neq i} \frac{P_{ik}(h)P_{kj}(t)}{h} - \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} P_{ij}(t)$$

If the sum in the first term is infinite, we need to justify why limit and summation commute. If they commute, we have

$$\frac{d}{dt} P_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

This differential equation is the backward Kolmogorov equation. Now we need to justify the commutation of limit and summation. For fixed N, we have

$$\liminf_{h \to 0} \sum_{k \neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) \geq \liminf_{h \to 0} \sum_{k \neq i, k < N} \frac{P_{ik}(h)}{h} P_{kj}(t) = \sum_{k \neq i, k < N} q_{ik} P_{kj}(t)$$

and this holds for any N. As a result,

$$\liminf_{h \to 0} \sum_{k \neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) \geq \sum_{k \neq i} q_{ik} P_{kj}(t)$$

Now for $N > i$,

$$\begin{aligned}
\limsup_{h \to 0} \sum_{k \neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) &\leq \limsup_{h \to 0} \left[\sum_{k \neq i, k < N} \frac{P_{ik}(h)}{h} P_{kj}(t) + \sum_{k} \frac{P_{ik}(h)}{h} \right] \\
&= \limsup_{h \to 0} \left[\sum_{k \neq i, k < N} \frac{P_{ik}(h)}{h} P_{kj}(t) + \frac{1 - P_{ii}(h)}{h} - \sum_{k \neq i, k < N} \frac{P_{ik}(h)}{h} \right] \\
&= \sum_{k \neq i, k < N} q_{ik} P_{kj}(t) + v_i - \sum_{k \neq i, k < N} q_{ik}
\end{aligned}$$

If we let $N \to \infty$ and note $v_i = \sum_{k \neq i} q_{ik}$, the last two terms cancel and we showed that

$$\limsup_{h \to 0} \sum_{k \neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) \leq \sum_{k \neq i} q_{ik} P_{kj}(t)$$

which justifies out interchange of limit and summation.

The equation is called backward equation is because we use the fact that

$$P_{ij}(t+h) = \sum_{k} P(X(t+h) = j | X(0) = i, X(h) = k) P(X(h) = k | X(0) = i) = \sum_{k} P_{kj}(t) P_{ik}(h)$$

Similarly, we could have a forward equation:

$$P_{ij}(t+h) = \sum_k P(X(t+h)=j|X(0)=i, X(t)=k)P(X(t)=k|X(0)=i) = \sum_k P_{ik}(t)P_{kj}(h)$$

We can do the difference and try to get a differential equation:

$$\lim_{h\to 0} \frac{P_{ij}(t+h)-P_{ij}(t)}{h} = \lim_{h\to 0}\left[\sum_{k\neq j} P_{ik}(t)\frac{P_{kj}(h)}{h} - \frac{1-P_{ij}(h)}{h}P_{ij}(t)\right]$$

It turns out the summation and limit cannot interchange in this case. We need some regularity and we need some regularity conditions to make the summation and limit interchangable. But if the state number is finite, we can still interchange the limit and get

$$\frac{d}{dt}P_{ij}(t) = \sum_{k\neq j} q_{kj}P_{ik}(t) - v_j P_{ij}(t)$$

A compact way to write the forward and backward equations is to define a quantity

$$r_{ij} = \begin{cases} q_{ij} & \text{if } i\neq j \\ -v_i & \text{if } i=j \end{cases}$$

and the two equations become

$$\text{Backward Equation:} \quad \frac{d}{dt}P_{ij}(t) = \sum_k r_{ik}P_{kj}(t) \quad \Rightarrow P'(t) = RP(t)$$

$$\text{Forward Equation:} \quad \frac{d}{dt}P_{ij}(t) = \sum_k P_{ik}(t)r_{kj} \quad \Rightarrow P'(t) = P(t)R$$

where we also invoked the matrix equation in the last part. We can try to solve the equation. The backward equation would give $P(t) = e^{Rt}$. Here we would use the definition of the matrix exponential. The definition is

$$e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!}$$

and a natural question is whether the definition converge? The sum may not converge. We can consider the Cauchy sum

$$\left\|\sum_{n=m+1}^{m+k}\frac{A^n}{n!}\right\| \leq \sum_{n=m+1}^{\infty}\frac{\|A\|^n}{n!} \leq \sum_{n=m+1}^{\infty}\frac{\|A\|^n}{n!}$$

which goes to zero when $m\to\infty$.

Nov 4th, 2021

Semi-Markov Process

Theorem: Suppose the semi-Markov process is irreducible and T_{ii} has a non-lattice distribution with finite mean. Embedded Markov Chain $\{X_n, n\geq 1\}$ is positive recurrent. Then

$$P_i = \lim_{t\to\infty}(z(t)=i|z(0)=j) = \frac{\pi_i\mu_i}{\sum_j \pi_j\mu_j}$$

μ_i is the average amount of time $Z(t)$ stays at state i before making a transition. and π is the stationary probabilities associated with $\{X_n\}$.
proof: Define $Y_i(j)$ is the amount of time spent in state i during the j-th visit to state i. $N_i(m)$ is the number of visits to state i in the first m transitions. $P_{i=m}$ is the proportion of time in i during the first m transitions.

$$P_{i=m} = \frac{\sum_{j=1}^{N_i(m)} y_i(j)}{\sum_i \sum_{j=1}^{N_i(m)} y_i(j)} = \left(\frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} \frac{y_i(j)}{N_i(m)} \right) \left(\sum_i \frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} \frac{y_i(j)}{N_i(m)} \right)^{-1}$$

Since $N_i(m) \to \infty$ as $m \to \infty$ because state i is recurrent, then

$$\sum_{j=1}^{N_i(m)} \frac{Y_i(j)}{N_i(m)} \to \mu_i \quad \frac{N_i(m)}{m} \to \pi_i$$

As a result, we find that

$$P_{i=m} = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}$$

Limiting probabilities for continuous time MC
In this case,

$$P_j = \lim_{t\to\infty} P_{ij}(t) = \lim_{t\to} P(X(t) = j|X(0) = i)$$

here we have $\mu_i = 1/v_i$ and thus,

$$P_{ij} = \frac{\pi_j/v_j}{\sum_i \pi_i/v_i}$$

But we know that for stationary distribution

$$v_j p_j = \sum_i v_i p_i P_{ij} \quad \Rightarrow v_j p_j = \sum_i p_i q_{ij}$$

where the left hand side the number of jumps leaving state i and the right hand side the number of jumps into state i.
Remarks:

1. P_j is the long run proportion of time the process in state j.
2. Also, if the initial state is chosen according to the limiting probabilities P_j then the resulting process will be stationary.
3. $p_j = \sum_i P_{ij}(h)p_i = \sum_{i\neq j}(q_{ij}h+o(h))p_i+(1-v_jh+o(h))p_j$ Thus, we have $0 = \sum_{i\neq j} p_i q_{ij} - v_j P_j + o(h)/h$ which gives the balance equation
$$\sum_{i\neq j} p_i q_{ij} = v_j p_j$$

Continuous time MC is ergodic if it is irreducible and $P_j > 0$ for all j.
In the Birth-Death process, we have the following equations:

$$\lambda_0 p_0 = \mu_1 p_1$$
$$(\lambda_0 + \mu_0)P_n = \mu_{n+1}P_{n+1} + \lambda_{n-1}P_{n-1}$$

We find that $\lambda_n P_n = \mu_{n+1} P_{n+1}$ given $\lambda_0 P_0 = \mu_1 P_1$ and thus, we find

$$P_n = \frac{\lambda_{n-1} \dots \lambda_1 \lambda_0}{\mu_n \dots \mu_2 \mu 1} P_0$$

we can determine p_0 by the normalization condition.

Nov 9th, 2021

Brownian Motion

Standard Brownian motion or Wiener Process. $\{X(t), t \geq 0\}$ is Brownian Motion if

(1) $X(0) = 0$

(2) $X(t)$ has stationary and independent increments

(3) For every $t > 0$, $X(t) \sim N(0, t)$.

(4) $t \to X(t)$ is continuous (It can be derived from above 3)

We can get the covariance as

$$\text{Cov}[X(s), X(t)] = \text{Cov}[X(s), X(t) - X(s) + X(s)] = \text{Cov}[X(s), X(s)] = s$$

where we assumed $s < t$. Therefore, $\text{Cov}[X(s), X(t)] = s \wedge t \equiv \min(s, t)$

Property

1. Brownian motion is a limit of rescaled Random Walk. It is a random step function with jumps of size $1/\sqrt{n}$ at step k/n. Thus, its step size would be $\sqrt{\Delta t}$ with $\Delta t = 1/n$.

2. Let $W(t)$ be a Brownian Motion then each of the following is also a Brownian motion:

   (a) $\{-W(t)\}$
   (b) $\{W(t+s) - W(s)\}_{t \geq 0}$
   (c) $\{aW(t/a^2)\}$
   (d) $\{tW(1/t)\}$

3. The path of Brownian Motion is continuous but is nowhere differentiable.

4. It has Markovian property:

$$\begin{aligned}
&P(X(t+s) \leq a | X(s) = x, X(u), 0 \leq u < s) \\
&= P(X(t+s) - X(s) \leq a - s | X(s) = x, X(u)) \\
&= P(X(t+s) - X(s) \leq a - x) = P(X(t+s) \leq a | X(s) = x)
\end{aligned}$$

   Where we used the independent increment property. This is just Markovian.

5. The process is a Gaussian process.

   Gaussian Process A stochastic process $X(t)$ is Gaussian process, if $X(t_1), \dots, X(t_n)$ has a multivariate normal distribution.

   We can show that BM $W(t)$ is a Gaussian process by noting that

$$W(t_1), \dots, W(t_n) = M \dots (W(t_1) - W(0), W(t_2) - W(t-3), \dots)$$

   where we used the fact that $W(0) = 0$ and M is a linear map that does the desired mapping. Not the right hand side is a joint distribution for n independent Gaussians. Thus, the left hand side is a multivariate Gaussian.

Conditional Distribution of $X(s)|X(t) = B$, $s < t$.

$$\begin{aligned} f(X|B) &= \frac{f_s(x) f_{t-s}(B-x)}{f_t(B)} \\ &= \frac{e^{-x^2/2s} e^{-(B-x)^2/2(t-s)}}{e^{-t^2/2B}} \\ &= C(s,t) \exp\left(-t \frac{(x - BS/t)^2}{2s(t-s)}\right) \end{aligned}$$

Thus, The expectation is $E[X(s)|X(t) = B] = Bs/t$. Also, the variance is $\text{Var}(X(s)|X(t) = B) = s(t-s)/t$. The Process $\{X(t)|X(1) = 0, 0 \le t \le 1\}$ is called the Brownian bridge. The mean of the process is just $E[X(s)|X(1) = 0] = 0$. We can calculate the covariance

$$\text{Cov}[X(s), X(t)] = \mathbb{E}[X(s)X(t)|X(1) = 0] = \mathbb{E}[\mathbb{E}[X(s)X(t)|X(t), X(1) = 0]|X(1) = 0]$$

The inner conditional expectation is

$$E[X(s)X(t)|X(t), X(1)] = X(t)E[X(s)]|X(t), X(1)] = X(t)E[X(s)|X(t), X(1) - X(t)] = E[X(s)|X(t)]X(t)$$

This number is thus $X(t)\frac{s}{t}X(t)$ and thus the covariance is

$$\text{Cov}[X(s), X(t)] = \frac{s}{t}\mathbb{E}[X^2(t)|X(1) = 0] = \frac{s}{t}t(1-t) = s(1-t)$$

Thus the Brownian bridge is just a Gaussian process with mean zero and covariance function $s(1-t)$.

Propostion Suppose $X(t)$ is a Brownian motion. Then $Z(t) = X(t) - tX(1)$ is a Brownian Bridge with $t \in [0, 1]$.

Example: $X_1, \dots, X_n \sim U(0,1)$ iid. $N_n(s) = \sum_{i=1}^n I_i(s)$ where I_i is the indicator function for $X \le s$. Then $F_n(s) = N_n(s)/n$ would be the empirical cdf. Fix $s \in (0,1)$, Then $F_n(s) \to s$ with probability 1. Let $\alpha_n(s) = \sqrt{n}(F_n(s) - s)$ then, $\alpha_n(s)$ converges to Brownian Bridge process.

Application: $X_i \sim F$ and then $F(x_i) \sim U(0,1)$. We are interested in $\sqrt{n}\sup_x |F_n(x) - F(x)|$. Let

$$\begin{aligned} \alpha_n(s) &= \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^n I(F(X_i) \le s) - s\right] \\ &= \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^n I(X_i \le F^{-s}(s)) - s\right] \\ &= \sqrt{n}[F_n(F^{-1}(s) - s] \end{aligned}$$

Then we see that

$$\sqrt{n}\sup_x |F_n(s) - F(x)| = \sqrt{n}\sup_s |F_n(F^{-1}(s)) - s| = \sup_s |\alpha_n(s)|$$

Then this is just the supreme of a Brownian bridge process by the continuous mapping theorem. The supreme of Brownian bridge process has already been calculated and we can find tabulated values.

Nov 11th, 2021

$\{W(t+s) - W(s)\}_{t\geq 0}$ is a Brownian Motion and is independent of the path $\{W(r)\}_{r\in(0,s)}$.

Stopping time: a non-negative random variable τ is a stopping time for Brownian Motion $W(t)$ if for every $t \geq 0$, $\{\tau \leq t\}$ depends on the sequence $\{W(r)\}_{r\in[0,t]}$.

For example $\tau(a) = \inf\{t|W(t) = a\}$ is a stopping time.

Let $W^*(t) = W(t+\tau) - W(\tau)$, $t \geq 0$. τ is a stopping time. Then $W^*(t)$ is a Brownian Motion. and the process W^* is independent of the pre-τ process $\{W(s)\}_{s\leq\tau}$. The process is called a post-τ process.

Corollary: Let $\{W_s^*\}$ be a second BM in the same probability space that is independent of the stopping field $\mathcal{F}_\tau$. Then the **splicing**:

$$\tilde{W}_t = \begin{cases} W_t & t \leq \tau \\ W_\tau + W^*_{t-\tau} & t \geq \tau \end{cases}$$

is a Brownian Motion.

Now we have a hitting time τ and the BM $W^*(t) = W(t+\tau) - W(\tau)$ and W^* is independent of $W(s)$ with $s \leq \tau$. Then the splicing motion:

$$\tilde{W}(s) = \begin{cases} W(s) & s \leq \tau \\ W(\tau) + (-W^*) = 2a - W(s) & s \geq \tau \end{cases}$$

is a Brownian motion because $-W^*(s)$ is a Brownian Motion. This is called **reflection principle**. This implies: for each $W(t)$ path, we have a $\tilde{W}(t)$ path. Thus,

$$P(W(t) < a|\tau < t) = P(\tilde{W}(t) < a|\tau < t) = P(2a - W(t) < a|\tau < t) = P(W(t) > a|\tau < t)$$

We can then calculate the distribution $\tau_a \leq t$.

$$P(W(t) \geq a) = P(W(t) \geq a|\tau_a \leq t)P(\tau_a \leq t) + P(W(t) \geq a|\tau_a > t)P(\tau_a > t)$$

However, $P(W(t) \geq a|\tau_a > t) = 0$. Therefore, we find that

$$P(W(t) \geq a) = P(W(t) \geq a|\tau_a \leq t)P(\tau_a \leq t) = \frac{1}{2}P(\tau_a \leq t)$$
$$\Rightarrow P(\tau_a \leq t) = \frac{2}{\sqrt{2\pi t}}\int_a^\infty e^{-x^2/2t}dx = \frac{2}{\sqrt{2\pi}}\int_{a/\sqrt{t}}^\infty e^{-y^2/2}dy$$

let $t \to \infty$, we find $P(\tau_a < \infty) = \lim_{t\to\infty} P(\tau_a < t) = 1$. Thus, every a can be reached in a finite time with probability one. However we find $E[\tau_a] = \infty$.

In the above calculation, we assumed $a > 0$ in the conditional expansion and if $a < 0$, we can do similar calculation and find that

$$P(\tau_a \leq t) = P(\tau_{-a} \leq t) = \frac{2}{\sqrt{2\pi}}\int_{|a|/\sqrt{t}}^\infty e^{-y^2/2}dy$$

Then $P(M(t) \geq a) = P(\tau_a \leq t) = 2P(W(t) \geq a) = P(|W(t)| \geq a)$ which is quite interesting.
Let $O(t_1, t_2)$ be the evetns that BM takes on the value 0 at least once within the interval (t_1, t_2). Then

$$P(O(t_1, t_2)) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} P(O(t_1, t_2)|W(t_1)) = a) e^{-x^2/2t_1} dx$$

However, $P(O(t_1, t_2)|W(t_1) = x) = P(\tau_{|x|} \leq t_2 - t_1)$ which means, if the BM is at x at time t_1 then it is better to pass zero before time t_2. As a result, we have

$$P(O(t_1, t_2)) = \frac{1}{\pi\sqrt{t_1(t_2 - t_1)}} \int_0^{\infty} \int_x^{\infty} e^{-y^2/2(t_2 - t_1)} dy dx = 1 - \frac{2}{\pi} \arcsin\Big(\sqrt{(t_1/t_2)}\Big)$$

The Arcsin Distribution: A random vairable X on $[0, 1]$ in arsin distribution if

$$P(X \leq x) = \frac{2}{\pi} \arcsin\big(\sqrt{x}\big), \quad \forall x \in [0, 1]$$
$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad x \in (0, 1)$$

Three arcsin laws for Brownian Motion:

(1) A $=$ proportion of time that $W(t)$ stays positive

(2) $L = \sup\{t \in [0, 1]|W(t) = 0\}$ the time of last zero in $[0, 1]$.

(3) $T = \arg\max_{s \in [0,1]} W(s)$

The first arcsin law can be seen from the random walk calculation previously. Since Brownian motion is a continuous version of the random walk, this should not be a surprise.
To see the second law, we need to calculate

$$P(L \leq x) = P\{\text{no zero in } (x, 1]\} = \frac{2}{\pi} \arcsin\big(\sqrt{x}\big)$$

Because we know $P(O(x, 1))$ which is the events of at least visiting 0 once and we only need to use $1 - P(O(x, 1))$.
We now prove the third law.
proof : For any $t \in (0, 1)$, let

$$X_r = B_{t-r} - B_t, \quad 0 \leq r \leq t$$

and we can check X_r is a Brownian motion. We can also define

$$Y_s = B_{t+s} - B_t \quad 0 \leq s \leq 1 - t$$

Now the event that $T \leq t$ is equivalent to

$$\begin{aligned}
P(T \leq t) &= P(\max_{u \in [0,t]} B_u > \max_{u \in [t,1]} B_u) \\
&= P(\max_{u \in [0,t]} (B_u - B_t) > \max_{u \in [t,1]} (B_u - B_t)) \\
&= P(\max_{r \in [0,t]} (B_{t-r} - B_t) > \max_{s \in [0,1-t]} (B_{t+s} - B_t)) \\
&= P(\max_{r \in [0,t]} X_r > \max_{s \in [0,1-t]} Y_s)
\end{aligned}$$

But we know that $\max_{r\in[0,t]} X_r$ has the same distribution of $|X_t|$ and $\max_{s\in[0,1-t]} Y_s$ has the same distribution of $|Y_{t-1}|$. and $|X_t|$ has the same distribution of $\sqrt{t}|Z_1|$ and $|Y_{1-t}|$ has the same distribution of $\sqrt{1-t}|Z_2|$ with Z_1, Z_2 iid standard normal. Thus, we have

$$\begin{aligned} P(T \le t) &= P(\sqrt{t}|Z_1| > \sqrt{1-t}|Z_2|) \\ &= P\left(\frac{|Z_2|}{\sqrt{Z_1^2 + Z_2^2}} < t\right) \\ &= P(|\sin(\theta)| < \sqrt{t}) = P(-\sqrt{t} < \sin\theta < \sqrt{t}) = \frac{2}{\pi}\arcsin\Big(\sqrt{t}\Big) \end{aligned}$$

This shows the arcsin law distribution of the max occurring time.

Nov 18th

The zero sets of Brownian Motion is $Z = \{t|W_t = 0\}$.

$$P(Z \cap [1,t] \neq \phi) = P(W_s \text{ take on zero at least once in } [1,t]) = 1 - \frac{2}{\pi}\arcsin\Big(\sqrt{1/t}\Big)$$

Letting $t \to \infty$, the probability goes to 1. Thus, with probability 1. the Brownian motion eventually returns to the origin.

Let τ_1 be the first time it returns to zero in $[1,\infty)$. It is obvious from the previous conclusion that $\tau_1 < \infty$ almost surely. As a result, $\tau_i - \tau_{i-1} < \infty$ almost surely. Thus, Brownian Motion returns to zero infinitely often. We can infer that for large t, the BM has both positive and negative values.

As $t \to 0$, we know that $Y_t = tX_{1/t}$ is a Brownian motion. Since $1/t \to \infty$ as $t \to 0$ and thus $X_{1/t}$ has both negative and positive values. Thus, Y_t takes bboth positive and negative values for arbitrary small values of t.

Proposition: With probability one, the Brownian path $W(t)$ has infinitely many zeros in every interval $(0,\epsilon)$.

But what is the measure of the set Z, $|Z|$? We see that

$$E[|Z|] = E\int_0^\infty \mathbb{I}(W_t = 0)dt = \int_0^\infty P(W_t = 0)dt = 0$$

Because the probability of $W_t = 0$ for a specific t is zero.

Specifically, $Z = \{0 \le t \le 1|W(t) = 0\}$ is a one-to-one map to the Canter set. (The set is uncountable and has zero measure).

Embedded Simple Random Walk

Define $\tau = \min\{t||W_t| = 1\}$. and this is stopping time.

Define $\tau_0 = 0$ and $\tau_{n+1} = \inf\{t > \tau_n||W_t - W_{\tau_n}| = 1\}$. $Y_n = W(\tau_n)$ is a Symmetric Random Walk (SRW).

We know that $\tau < \infty$ almost surely. Since in SRW, each state is recurrent. Thus, the embedded SRW will visit every integer in finite number of steps. Since each step is finite in time and thus, each integer can be visited by the BM in a finite time almost surely.

Next we want to show that $\lim_{t\to\infty} X_t/t = 0$ almost surely.

We know that

$$\frac{X_n}{n} = \frac{\sum_{i=1}^n (X_j - X_{j-1})}{n} \to 0. \quad \text{a.s}$$

by strong law of large numbers. But here we want a continuous version.
We denote $M_n = \sup\{X_t - X_n, n \le t < n+1\}$. But we know that

$$\frac{X_t}{n} \le \frac{X_n}{n} + \frac{X_t - X_n}{n} \le \frac{X_n}{n} + \frac{M_n}{n}$$

We can try to show $M_n/n \to \infty$.

$$\begin{aligned} P(M_n \ge a) &= P(\sup_{0\le s\le 1} W_s \ge a) = P(|W_1| \ge a) \\ &= 2\int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \le 2\int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-xa/2} dx \\ &= \frac{4}{\sqrt{2\pi}a} e^{-a^2/2} \end{aligned}$$

Take $a = 2(\ln(n))^{1/2}$ we have

$$P(M_n \ge 2(\ln(n))^{1/n}) \le \frac{2}{\sqrt{2\pi}\sqrt{\ln n}} n^{-2}$$

By Borel Cantelli lemma:

$$E\left(\sum_{n=1}^{\infty} I(M_n \ge 2\ln(n)^{1/2})\right) = \sum_{n=1}^{\infty} \frac{2}{\sqrt{2\pi}\sqrt{\ln n}} n^{-2} < \infty$$

Thus, $M_n \ge ln(n)$ infinitely often has zero probability. We find that $M_n/n \to 0$ with probability 1.